Solutions to Question Sheet 8, Differentiation III. v1 2019-20

Taylor Polynomials

If f has n derivatives at $a \in \mathbb{R}$ then

$$T_{n,a}f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(x-a)^r .$$

There are four questions asking you to calculating Taylor polynomials and they all highlight a method that should simplify the work needed and cut down the opportunity of making an error.

1. Calculate the Taylor polynomial

$$T_{6,0}\left(\frac{\sin x + \cos x}{1+x}\right).$$

Hint Multiply up so you don't have to differentiate rational functions.

Solution Let

$$f(x) = \frac{\sin x + \cos x}{1+x}.$$

We **don't** like fractions so multiply up and consider

$$(1+x)\, f(x) = \sin x + \cos x.$$

We will differentiate this repeatedly to get

$$\begin{aligned}
(1+x)\, f^{(1)}(x) + f(x) &= \cos x - \sin x \\
(1+x)\, f^{(2)}(x) + 2f^{(1)}(x) &= -\sin x - \cos x \\
(1+x)\, f^{(3)}(x) + 3f^{(2)}(x) &= -\cos x + \sin x \\
(1+x)\, f^{(4)}(x) + 4f^{(3)}(x) &= \sin x + \cos x \\
(1+x)\, f^{(5)}(x) + 5f^{(4)}(x) &= \cos x - \sin x \\
(1+x)\, f^{(6)}(x) + 6f^{(5)}(x) &= -\sin x - \cos x.
\end{aligned}$$

Put $x = 0$ to get

$$\begin{aligned} f(0) &= 1 \\ f^{(1)}(0) + f(0) &= 1 \\ f^{(2)}(0) + 2f^{(1)}(0) &= -1 \\ f^{(3)}(0) + 3f^{(2)}(0) &= -1 \\ f^{(4)}(0) + 4f^{(3)}(0) &= 1 \\ f^{(5)}(0) + 5f^{(4)}(0) &= 1 \\ f^{(6)}(0) + 6f^{(5)}(0) &= -1. \end{aligned}$$

Solving these we find $f(0) = 1$, $f^{(1)}(0) = 0$, $f^{(2)}(0) = -1$, $f^{(3)}(0) = 2$, $f^{(4)}(0) = -7$, $f^{(5)}(0) = 36$ and $f^{(6)}(0) = -217$.

Hence

$$\begin{aligned} T_{6,0}\left(\frac{\sin x + \cos x}{1 + x}\right) &= 1 + 0x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 7\frac{x^4}{4!} + 36\frac{x^5}{5!} - 217\frac{x^6}{6!} \\ &= 1 - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 7\frac{x^4}{4!} + 36\frac{x^5}{5!} - 217\frac{x^6}{6!}. \end{aligned}$$

2. Calculate the Taylor polynomial

$$T_{8,0}\left(\sin x \cosh x\right).$$

Hint Look for a pattern in your derivatives. For the trigonometric functions $\sin x$ and $\cos x$ you return to a function related to the original function after differentiation at most 4 times. For hyperbolic functions it is after 2 differentiations. Thus for f that are products of such functions you might hope to see some connection between f and $f^{(4)}$.

Solution Let $f(x) = \sin x \cosh x$. Then

$$\begin{aligned}
f^{(1)}(x) &= \cos x \cosh x + \sin x \sinh x, \\
f^{(2)}(x) &= -\sin x \cosh x + \cos x \sinh x \\
&\quad + \cos x \sinh x + \sin x \cosh x \\
&= 2 \cos x \sinh x, \\
f^{(3)}(x) &= -2 \sin x \sinh x + 2 \cos x \cosh x, \\
f^{(4)}(x) &= -2 \cos x \sinh x - 2 \sin x \cosh x \\
&\quad -2 \sin x \cosh x + 2 \cos x \sinh x \\
&= -4 \sin x \cosh x = -4 f(x) \,.
\end{aligned}$$

From $f^{(4)}(x) = -4f(x)$ we quickly get

$$\begin{aligned}
f^{(5)}(x) &= -4 f^{(1)}(x) \,, \\
f^{(6)}(x) &= -4 f^{(2)}(x) \,, \\
f^{(7)}(x) &= -4 f^{(3)}(x) \,, \\
f^{(8)}(x) &= -4 f^{(4)}(x) = 16 f(x) \,.
\end{aligned}$$

Hence $f(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = 2$, $f^{(4)}(0) = 0$, $f^{(5)}(0) = -4$, $f^{(6)}(0) = 0$, $f^{(7)}(0) = -8$ and $f^{(8)}(0) = 0$. Thus

$$\begin{aligned}
T_{8,0}\left(\sin x \cosh x\right) &= 0 + 1x + 0x^2 + \frac{1}{3}x^3 + 0x^4 - \frac{1}{30}x^5 + 0x^6 - \frac{1}{630}x^7 + 0x^8 \\
&= x + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{630}x^7.
\end{aligned}$$

3. Calculate the Taylor polynomial

$$T_{5,0}\left(e^{\sin x}\right).$$

Hint Let $f(x) = e^{\sin x}$ and, because of the exponential function satisfies $de^x/dx = e^x$, look for a connection between f and $f^{(1)}$.

Solution Let $f(x) = e^{\sin x}$. Then by the Composition Rule for differentiation

$$f^{(1)}(x) = e^{\sin x} \cos x = f(x) \cos x.$$

Thus

$$\begin{aligned}
f^{(2)}(x) &= f^{(1)}(x)\cos x - f(x)\sin x, \\
f^{(3)}(x) &= f^{(2)}(x)\cos x - f^{(1)}(x)\sin x - f^{(1)}(x)\sin x - f(x)\cos x \\
&= f^{(2)}(x)\cos x - 2f^{(1)}(x)\sin x - f(x)\cos x, \\
f^{(4)}(x) &= f^{(3)}(x)\cos x - f^{(2)}(x)\sin x - 2f^{(2)}(x)\sin x \\
& \quad -2f^{(1)}(x)\cos x - f^{(1)}(x)\cos x + f(x)\sin x \\
&= f^{(3)}(x)\cos x - 3f^{(2)}(x)\sin x - 3f^{(1)}(x)\cos x + f(x)\sin x
\end{aligned}$$

Hopefully you can see a pattern (reminiscent of the Binomial Theorem?) and the next in the list will be

$$f^{(5)}(x) = f^{(4)}(x)\cos x - 4f^{(3)}(x)\sin x - 6f^{(2)}(x)\cos x + 4f^{(1)}(x)\sin x + f(x)\cos x.$$

Putting $x = 0$ and we find $f(0) = 1, f'(0) = 1$ and

$$\begin{aligned}
f^{(2)}(0) &= f^{(1)}(0) = 1 \\
f^{(3)}(0) &= f^{(2)}(0) - f(0) = 0 \\
f^{(4)}(0) &= f^{(3)}(0) - 3f^{(1)}(0) = -3, \\
f^{(5)}(0) &= f^{(4)}(0) - 6f^{(2)}(0) + f(0) = -3 - 6 + 1 = -8.
\end{aligned}$$

Thus

$$T_{5,0}\left(e^{\sin x}\right) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15}.$$

4. Calculate the Taylor Polynomial

$$T_{4,0}\left(\frac{\ln(1+x)}{1+x}\right).$$

Hint Again look at multiplying up and writing a derivative in terms of earlier derivatives.

Solution Let

$$f(x) = \frac{\ln(1+x)}{1+x}.$$

Follow the hint and write $(1+x)\,f(x) = \ln(1+x)$. Then, taking the derivative,

$$f(x) + (1+x)\,f^{(1)}(x) = \frac{1}{1+x}.$$

Follow the hint yet again, and multiply up as

$$(1+x)\,f(x) + (1+x)^2\,f^{(1)}(x) = 1.$$

Repeated differentiation gives

$$\begin{aligned} f(x) + 3\,(1+x)\,f^{(1)}(x) + (1+x)^2\,f^{(2)}(x) &= 0, \\ 4f^{(1)}(x) + 5\,(1+x)\,f^{(2)}(x) + (1+x)^2\,f^{(3)}(x) &= 0, \\ 9f^{(2)}(x) + 7\,(1+x)\,f^{(3)}(x) + (1+x)^2\,f^{(4)}(x) &= 0. \end{aligned}$$

Substituting $x = 0$ gives

$$\begin{aligned} f(0) + f^{(1)}(0) &= 1, \\ f(0) + 3f^{(1)}(0) + f^{(2)}(0) &= 0, \\ 4f^{(1)}(0) + 5f^{(2)}(0) + f^{(3)}(0) &= 0, \\ 9f^{(2)}(0) + 7f^{(3)}(0) + f^{(4)}(0) &= 0. \end{aligned}$$

Starting with $f(0) = 0$ we get $f^{(1)}(0) = 1$, $f^{(2)}(0) = -3$, $f^{(3)}(0) = -4 + 15 = 11$ and $f^{(4)}(0) = 27 - 77 = -50$.

Hence

$$\begin{aligned} T_{4,0}\left(\frac{\ln(1+x)}{1+x}\right) &= 0 + x - 3\frac{x^2}{2!} + 11\frac{x^3}{3!} - 50\frac{x^4}{4!} \\ &= x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4. \end{aligned}$$

Error Terms

The Remainder or Error Term in approximating a function by it's Taylor Polynomial is given by

$$R_{n,a}f(x) = f(x) - T_{n,a}f(x)\,.$$

In the notes we give bounds on $R_{n,a}f(x)$ which thus tell us how well $T_{n,a}f(x)$ approximates $f(x)$. This is the subject of the next three questions. But we can also deduce something from knowing that $R_{n,a}f(x)$ is of constant sign as x varies; we get inequalities between $f(x)$ and $T_{n,a}f(x)$.

5. i. Prove that

$$\left|\sin x - x + \frac{x^3}{6}\right| \le \frac{1}{4!}\left|x\right|^4, \tag{1}$$

for all $x \in \mathbb{R}$.

Hint the left hand side is $|R_{3,0}(\sin x)|$.

ii. Deduce (without L'Hôpital's Rule) that

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

Solution i. It is easy to check that

$$T_{3,0}(\sin x) = x - \frac{x^3}{3!},$$

so

$$\sin x - x + \frac{x^3}{3!} = \sin x - T_{3,0}(\sin x) = R_{3,0}(\sin x)\,.$$

Let $f(x) = \sin x$, then Lagrange's form of the error gives

$$R_{3,0}(\sin x) = \frac{f^{(4)}(c)}{4!}x^4,$$

for some c between x and 0. Yet $\left|f^{(4)}(c)\right| = |\sin c| \le 1$, giving the stated resultm (1).

ii. Dividing through the stated result by $|x|^3$ gives

$$\left| \frac{\sin x - x}{x^3} + \frac{1}{3!} \right| \leq \frac{1}{4!} |x| \, .$$

Let $x \to 0$ and we get

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6},$$

by quoting the Sandwich Rule.

6. For $f(x) = \ln(1+x)$, find the Taylor polynomial $T_{5,0}f(x)$ and calculate $T_{5,0}f(0.2)$.

   Use Lagrange's form of the error for the remainder to estimate the error in using $T_{5,0}f(0.2)$ to calculate $\ln 1.2$.

   Hence show that

$$0.18232000... < \ln 1.2 < 0.18232709... \, .$$

Solution Repeated differentiation gives us

$$\begin{aligned} f(x) &= \ln(1+x)\,, & f(0) &= 0, \\ f^{(1)}(x) &= \frac{1}{1+x}, & f^{(1)}(0) &= 1, \\ f^{(2)}(x) &= -\frac{1}{(1+x)^2}, & f^{(2)}(0) &= -1, \\ f^{(3)}(x) &= \frac{2}{(1+x)^3}, & f^{(3)}(0) &= 2, \\ f^{(4)}(x) &= -\frac{6}{(1+x)^4}, & f^{(4)}(0) &= -6, \\ f^{(5)}(x) &= \frac{24}{(1+x)^5}, & f^{(5)}(0) &= 24. \end{aligned}$$

Thus

$$T_{5,0}f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

The approximation to $f(0.2) = \ln 1.2$ given by this polynomial is

$$\begin{aligned} T_{5,0}f(0.2) &= 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} \\ &= 0.18233066... \,. \end{aligned}$$

Also $f^{(6)}(x) = -5!/(1+x)^6$ and so Lagrange's form of the error is

$$R_{5,0}f(x) = \frac{-x^6}{6(1+c)^6},$$

for some c between 0 and x. With $x = 0.2$ then $1/1.2 < 1/(1+c) < 1$ and thus

$$-\frac{(0.2)^6}{6} < R_{5,0}f(0.2) < -\frac{(0.2)^6}{6(1.2)^6}.$$

That is,

$$-0.00001066... < \ln 1.2 - T_{5,0}f(0.2) < -0.0000035722... \,.$$

Hence

$$0.18232000... < \ln 1.2 < 0.18232709... \,.$$

In fact $\ln 1.2 = 0182321556....$

7. Use Taylor's Theorem with $f(x) = \sqrt{x}$ on $[64, 66]$ and $n = 1$ along with *Lagrange*'s form of the error to show that

$$\frac{1}{8} - \frac{1}{1024} < \sqrt{66} - 8 < \frac{1}{8} - \frac{1}{1458}.$$

Solution With $f(x) = \sqrt{x}$, $n = 1$, $a = 64$ and Lagrange's form of the error, Taylor's Theorem states

$$R_{1,64}f(x) = \frac{f^{(2)}(c)}{2!}(x-64)^2,$$

for some c between 64 and x. That is

$$f(x) - T_{1,64}f(x) = \frac{f^{(2)}(c)}{2!}(x-64)^2,$$

or

$$f(x) = f(64) + f^{(1)}(64)(x-64) + \frac{f^{(2)}(c)}{2!}(x-64)^2.$$

With $f(x) = \sqrt{x}$ we get

$$\sqrt{x} = \sqrt{64} + \frac{(x-64)}{2\sqrt{64}} - \frac{(x-64)^2}{8c^{3/2}}.$$

Take $x = 66$ when $64 < c < 66$ and

$$\sqrt{66} - \sqrt{64} = \frac{(66-64)}{2\sqrt{64}} - \frac{(66-64)^2}{8c^{3/2}} = \frac{1}{8} - \frac{1}{2c^{3/2}}.$$

To simplify matters think of c as lying between 64 and 81 (the smallest square larger than 66), so

$$\frac{1}{1458} < \frac{1}{2c^{3/2}} < \frac{1}{1024}.$$

Thus

$$\frac{1}{8} - \frac{1}{1024} < \sqrt{66} - \sqrt{64} < \frac{1}{8} - \frac{1}{1458}.$$

In fact,

$$\sqrt{66} - 8 = \frac{1}{8} - \frac{1}{1039.938...}$$

Taylor Series

8. Calculate the Taylor Series for $x\cosh x + \sinh x$ with $a = 0$.

   Solution i) Let $f(x) = x\cosh x + \sinh x$. Then

$$\begin{aligned}
f^{(1)}(x) &= x\sinh x + 2\cosh x \\
f^{(2)}(x) &= x\cosh x + 3\sinh x \\
f^{(3)}(x) &= x\sinh x + 4\cosh x \\
f^{(4)}(x) &= x\cosh x + 5\sinh x \\
f^{(5)}(x) &= x\sinh x + 6\cosh x \\
&\vdots
\end{aligned}$$

The pattern is

$$f^{(r)}(x) = \begin{cases} x\sinh x + (r+1)\cosh x & \text{if } r \text{ is odd} \\ x\cosh x + (r+1)\sinh x & \text{if } r \text{ is even.} \end{cases}$$

Thus

$$f^{(r)}(0) = \begin{cases} (r+1) & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

Hence the Taylor Series for $x\cosh x + \sinh x$ with $a = 0$ is

$$\sum_{r=0}^{\infty} f^{(r)}(0)\,\frac{x^r}{r!} = \sum_{\substack{r=0 \\ r \text{ odd}}}^{\infty} (r+1)\,\frac{x^r}{r!} = \sum_{n=0}^{\infty} \frac{2(n+1)}{(2n+1)!} x^{2n+1}.$$

The first few terms are

$$2x + \frac{2}{3}x^3 + \frac{1}{20}x^5 + \frac{1}{630}x^7 + \frac{1}{36288}x^9 + ...$$

9. Prove that the Taylor series for cosine converges to $\cos x$, i.e.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!},$$

for all $x \in \mathbb{R}$.

Solution Let $f(x) = \cos x$. Let $x \in \mathbb{R}$ be given. Then for $n \geq 1$

$$R_{n,0}(\cos x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between x and 0. Yet $\left|f^{(n+1)}(c)\right|$ is either $|\sin c|$ or $|\cos c|$, both of which are ≤ 1. Thus

$$|R_{n,0}(\cos x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$, since $\left\{|x|^{n+1} / (n+1)!\right\}_{n\geq 1}$ is a null sequence. Hence $R_{n,0}(\cos x) \to 0$ as $n \to \infty$ and so the Taylor series for cosine converges to $\cos x$ for all $x \in \mathbb{R}$.

Additional Questions

10. Assume the function f is $n+1$ times differentiable with $f^{(n+1)}$ continuous on an open interval containing $a \in \mathbb{R}$. Prove that

$$\lim_{x\to a}\frac{f(x)-T_{n,a}f(x)}{(x-a)^n}=0 \quad \text{and} \quad \lim_{x\to a}\frac{f(x)-T_{n,a}f(x)}{(x-a)^{n+1}}=\frac{f^{(n+1)}(a)}{(n+1)!}. \tag{2}$$

Hint Consider Lagrange's error.

Note these limits in special cases have been seen many times before.

(a) $f(x)=\sin x$ with $T_{2,0}(\sin x)=x$ is the subject of Question 5,

(b) $f(x)=e^x$ with $T_{3,0}(e^x)=1+x+x^2/2$ is the subject of Question 9 on Sheet 3.

(c) $f(x)=\sinh x$ with $T_{2,0}(\sinh x)=x$ is the subject of the same question. To check that earlier answer

$$\begin{aligned}\lim_{x\to 0}\frac{\sinh x - x}{x^3} &= \lim_{x\to 0}\frac{\sinh x - T_{2,0}(\sinh x)}{x^3}\\ &= \frac{1}{3!}\frac{d^3}{dx^3}(\sinh x)\bigg|_{x=0} \quad \text{by (2)}\\ &= \frac{1}{6}.\end{aligned}$$

Solution For x lying in the interval around a in which f has $n+1$ derivatives Lagrange's error states that

$$f(x)-T_{n,a}f(x)=R_{n,a}f(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c lying between a and x. Therefore, for $x\neq a$,

$$\frac{f(x)-T_{n,a}f(x)}{(x-a)^{n+1}}=\frac{f^{(n+1)}(c)}{(n+1)!}. \tag{3}$$

Let $x\to a$. Since c lies between a and x we also have $c\to a$. We are assuming $f^{(n+1)}$ is continuous at a so $\lim_{c\to a} f^{(n+1)}(c)=f^{(n+1)}(a)$. Hence

$$\lim_{x\to a}\frac{f(x)-T_{n,a}f(x)}{(x-a)^{n+1}}=\frac{f^{(n+1)}(a)}{(n+1)!}.$$

Then, rearranging (3),

$$\frac{f(x) - T_{n,a}f(x)}{(x-a)^n} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a) \to 0$$

as $x \to a$, having used the Product Rule for limits, allowable since each individual limit exists.

11. i. Prove that $x^{n+1}R_{n,0}(e^x) \geq 0$ for all $x \in \mathbb{R}$.

Deduce that for all $m \geq 1$ we have

$$e^x \geq T_{2m-1,0}(e^x)$$

for all $x \in \mathbb{R}$, while

$$\begin{cases} e^x \geq T_{2m,0}(e^x) & \text{for } x > 0 \\ e^x \leq T_{2m,0}(e^x) & \text{for } x < 0. \end{cases}$$

Note this answers a question in the printed lecture notes, of showing that

$$e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

for all $x \in \mathbb{R}$ while

$$e^x > 1 + x + \frac{x^2}{2} \text{ if } x > 0 \quad \text{and} \quad e^x < 1 + x + \frac{x^2}{2} \text{ if } x < 0.$$

ii. Prove that $(-1)^n x^{n+1} R_{n,0}(\ln(1+x)) \geq 0$ for all $x > -1$.

Deduce that for all $n \geq 1$,

$$\ln(1+x) \leq T_{n,0}(\ln(1+x))$$

for $-1 < x < 0$ while if $x > 0$ then

$$\begin{cases} \ln(1+x) \leq T_{n,0}(\ln(1+x)) & \text{for odd } n \\ \ln(1+x) \geq T_{n,0}(\ln(1+x)) & \text{for even } n. \end{cases}$$

Note These last results for $x > 0$ can be combined in

$$T_{2m,0}\left(\ln\left(1+x\right)\right) \leq \ln\left(1+x\right) \leq T_{2m+1,0}\left(\ln\left(1+x\right)\right)$$

for all $m \geq 1$. The case $m = 1$ is the content of Question 6, Sheet 7.

Solution i. For all $x \in \mathbb{R}$ there exists, by Lagrange's form of the error term, some c between 0 and x such that

$$R_{n,0}\left(e^x\right) = \frac{e^c}{(n+1)!}x^{n+1}.$$

Then

$$x^{n+1}R_{n,0}\left(e^x\right) = \frac{e^c}{(n+1)!}\left(x^2\right)^{n+1} \geq 0, \tag{4}$$

for all $x \in \mathbb{R}$, since $e^c > 0$ for all c.

There are two cases.

If n is odd then $n+1$ is even so $x^{n+1} \geq 0$ for all x. Thus, by (4), $R_{n,0}\left(e^x\right) \geq 0$. Writing $n = 2m-1$ this implies $e^x \geq T_{2m-1,0}\left(e^x\right)$ for all $x \in \mathbb{R}$.

If n is even then $x^{n+1} \geq 0$ for all $x > 0$ and $x^{n+1} \leq 0$ for all $x < 0$. Thus, by (4), $R_{n,0}\left(e^x\right) \geq 0$ if $x > 0$ and $R_{n,0}\left(e^x\right) \leq 0$ if $x < 0$. Writing $n = 2m$ this implies $e^x \geq T_{2m,0}\left(e^x\right)$ for $x > 0$ and $e^x \leq T_{2m,0}\left(e^x\right)$ for $x < 0$.

ii. For all $x \in \mathbb{R}$ there exists, by Lagrange's form of the error term, some c between 0 and x such that

$$R_{n,0}\left(\ln\left(1+x\right)\right) = \frac{\left(-1\right)^n x^{n+1}}{(n+1)\left(1+c\right)^{n+1}}.$$

Then

$$\left(-1\right)^n x^{n+1}R_{n,0}\left(\ln\left(1+x\right)\right) = \frac{\left(x^2\right)^{n+1}}{(n+1)\left(1+c\right)^{n+1}} \geq 0 \tag{5}$$

for all $x > -1$, since $1+c > 0$ for $c > x > -1$..

There are two cases.

If n is odd then (5) implies $x^{n+1}R_{n,0}\left(\ln\left(1+x\right)\right) \leq 0$ for $x > -1$. Again $x^{n+1} \geq 0$ for all x so $R_{n,0}\left(\ln\left(1+x\right)\right) \leq 0$ for $x > -1$.

Writing $n = 2m - 1$ this implies

$$\ln(1+x) \leq T_{2m-1,0}(\ln(1+x))$$

for $x > -1$.

If n is even then (5) implies $x^{n+1}R_{n,0}(\ln(1+x)) \geq 0$. As in Part i, $x^{n+1} \geq 0$ for all $x > 0$ and $x^{n+1} \leq 0$ for all $-1 < x < 0$. Writing $n = 2m$ these imply

$$\begin{aligned} R_{2m,0}(\ln(1+x)) &\geq 0 \quad \text{for} \quad x > 0, \\ R_{2m,0}(\ln(1+x)) &\leq 0 \quad \text{for} \quad -1 < x < 0. \end{aligned}$$

That is,

$$\ln(1+x) \geq T_{2m-1,0}(\ln(1+x))$$

for $x > 0$ and

$$\ln(1+x) \leq T_{2m-1,0}(\ln(1+x))$$

for $-1 < x < 0$. These results can be combined in the way described in the question.